Efficient Decomposition of Bimatrix Games

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Overview

We are interested in (doable) operations $\circ$ on the class of bimatrix games, such that

1. from Nash equilibria of games $G_1$ and $G_2$, we can obtain a Nash equilibrium of $G_1 \circ G_2$,
2. and from any Nash equilibrium of $G_1 \circ G_2$ we can obtain Nash equilibria of both $G_1$ and $G_2$.
3. Furthermore, $\circ^{-1}$ is doable, too.

then we may conclude:

- Solving a fixed finite number of games is no harder than solving a single game.
- Solving a large decomposable game is no harder than solving a number of small games.
Outline

The product-operator

The sum-operator

A detour on wtt-reductions

The decomposition algorithm
Defining products

Definition
Given an $n_1 \times m_1$ bimatrix game $(A^1, B^1)$ and an $n_2 \times m_2$ bimatrix game $(A^2, B^2)$, we define the $(n_1 n_2) \times (m_1 m_2)$ product game $(A^1, B^1) \times (A^2, B^2)$ as $(A, B)$ with $A_{[i_1,i_2][j_1,j_2]} = A^1_{i_1,j_1} + A^2_{i_2,j_2}$ and $B_{[i_1,i_2][j_1,j_2]} = B^1_{i_1,j_1} + B^2_{i_2,j_2}$.

\[
\begin{pmatrix}
A^1 + A^2_{11} & A^1 + A^2_{12} & \cdots \\
A^1 + A^2_{21} & A^1 + A^2_{22} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]
Results for equilibria

Theorem
If \((x^k, y^k)\) is a Nash equilibrium of \((A^k, B^k)\) for both \(k \in \{1, 2\}\), then \((x, y)\) is a Nash equilibrium of \((A, B)\), where \(x_{[i_1 i_2]} = x^1_{i_1} x^2_{i_2}\) and \(y_{[m_1 m_2]} = y^1_{m_1} y^2_{m_2}\).

Theorem
If \((x, y)\) is a Nash equilibrium of \((A, B)\), then \((x^1, y^1)\) given by
\[
x^1_i = \sum_{l=1}^{n_2} x_{[i, l]} \quad \text{and} \quad y^1_j = \sum_{l=1}^{m_2} y_{[j, l]}\]
is a Nash equilibrium of \((A^1, B^1)\).
The definition

Definition
Given an $n_1 \times m_1$ bimatrix game $(A^1, B^1)$ and an $n_2 \times m_2$ bimatrix game $(A^2, B^2)$, we define the $(n_1 + n_2) \times (m_1 + m_2)$ sum game $(A^1, B^1) + (A^2, B^2)$ via the constant $K > \max_{i,j}\{|A_{i,j}, B_{i,j}|\}$ as $(A, B)$ with:

\[
A_{i,j} = \begin{cases} 
A^1_{i,j} & \text{if } i \leq n_1, j \leq m_1 \\
A^2_{(i-n_1),(j-m_1)} & \text{if } i > n_1, j > m_1 \\
K & \text{otherwise}
\end{cases}
\]

\[
B_{i,j} = \begin{cases} 
B^1_{i,j} & \text{if } i \leq n_1, j \leq m_1 \\
B^2_{(i-n_1),(j-m_1)} & \text{if } i > n_1, j > m_1 \\
-K & \text{otherwise}
\end{cases}
\]

\[
\begin{pmatrix}
A^1 & K \\
K & A^2
\end{pmatrix}
\begin{pmatrix}
B^1 & -K \\
-K & B^2
\end{pmatrix}
\]
Lemma
Let \((x, y)\) be a Nash equilibrium of \((A^1, B^1) + (A^2, B^2)\). Then \(0 < \left(\sum_{i=1}^{n_1} x_i\right) < 1\) and \(0 < \left(\sum_{j=1}^{m_1} y_j\right) < 1\).

Theorem
If \((x, y)\) is a Nash equilibrium of \((A^1, B^1) + (A^2, B^2)\), then a Nash equilibrium \((x^1, y^1)\) of \((A^1, B^1)\) can be obtained as \(x^1_i = \frac{x_i}{\sum_{l=1}^{n_1} x_l}\) and \(y^1_j = \frac{y_i}{\sum_{l=1}^{m_1} y_l}\).
The second result on equilibria

Theorem
Let \((x^k, y^k)\) be a Nash equilibrium of \((A^k, B^k)\) resulting in payoffs \((P^k, Q^k)\) for both \(k \in \{1, 2\}\). Then \((x, y)\) is a Nash equilibrium of \((A^1, B^1) + (A^2, B^2)\), where
\[
x_i = x^1_i \frac{K - Q^2}{2K - Q^1 - Q^2} \quad \text{for} \quad i \leq n_1,
\]
\[
x_i = x^2_i \frac{K - Q^1}{2K - Q^1 - Q^2} \quad \text{for} \quad i > n_1,
\]
\[
y_j = y^1_j \frac{K - P^2}{2K - P^1 - P^2} \quad \text{for} \quad j \leq m_1,
\]
\[
y_j = y^2_j \frac{K - P^1}{2K - P^1 - P^2} \quad \text{for} \quad j > m_1.
\]
Complexity of the translations

The operations on games and Nash equilibria are very simple. In particular, they are

- computable even for real payoffs.
- BSS computable.
- polynomial-time computable for rational payoffs.
The product was originally introduced to show that $\text{Nash} \equiv^W \text{Nash}^*$, i.e. that computable many-one reductions and computable wtt reductions to the problem of finding a Nash equilibrium for a bimatrix game with real-valued payoffs coincide.

Similarly, the sum can be used to show polynomial-time many-one and polynomial-time wtt reductions to finding Nash equilibria of bimatrix games with rational payoffs coincide. This implies that PPAD is closed under polynomial-time wtt reductions.
The algorithm

Our basic algorithm proceeds as follows: To solve a game $(A, B)$

1. test whether $(A, B)$ is the sum of $(A^1, B^1)$ and $(A^2, B^2)$ via some constant $K$. If yes, solve $(A^1, B^1)$ and $(A^2, B^2)$ and combine their Nash equilibria to an equilibrium of $(A, B)$ via Theorem 7. If no,

2. test whether $(A, B)$ is the product of $(A^1, B^1)$ and $(A^2, B^2)$. If yes, solve $(A^1, B^1)$ and $(A^2, B^2)$ and combine their Nash equilibria to an equilibrium of $(A, B)$ via Theorem 2. If no,

3. find a Nash equilibrium of $(A, B)$ by some other means (i.e. the GAMBIT library).

**Runtime:** $O(S^2f(\lambda))$
Experimental procedure

1. We randomly generated a decomposition tree
2. and payoffs in the leaves of size $\leq 3 \times 3$,
3. such that the entire game would have size $(100 - -110) \times (100 - -1110)$.
4. Then we forgot about the structure, and only consider the resulting bimatrix game.
Some experimental results

Figure: gambit-gnm
Questions?

Thanks for listening.