Two Themes in Modal Logic

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Overview

Introduction

Reasoning about Local Properties
- Preliminaries
- Axiomatization
- The general case

Succinctness
- Formula Size Games
- Results

Conclusion

Modal Logic

- has become *the* logic for Knowledge Representation and Reasoning in AI
- useful to represent knowledge (K) belief (B), desires (D) and also temporal ($\Box$, $\bigcirc$) and cooperation ($\langle\langle C\rangle\rangle$) modalities
- provides a powerful syntax and a natural semantics: Kripke models

This Talk

- Modal and first order properties on frames are linked since work on correspondence theory ('76)
- This characterises global properties on Kripke models.
- In this talk, I will present one way to localise this.

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- In this talk, I will present one way to localise this.

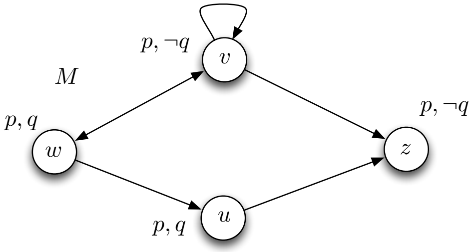
- One often sees similar modal logics for the same purpose.
- How to compare them?
- Typically: relative expressiveness, or complexity
- In this talk, I compare some on their succinctness

Truth

- Kripke model is $M = \langle W, R_i, V \rangle$
- define $M, s \models \varphi$, (local truth)
- then $M \models \varphi$ (model validity)
- then $\mathcal{C} \models \varphi$ (global validity)

A Model

$$M = \langle W, R, V \rangle$$



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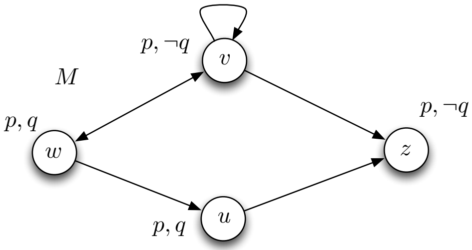


$$M, w \models (p \leftrightarrow q) \wedge \Box p \wedge \neg\Box q \wedge \Diamond(p \wedge \neg q) \wedge \Diamond\Box p$$

Correspondence

Sometimes: $C \models \Phi$ iff $C \models \varphi$

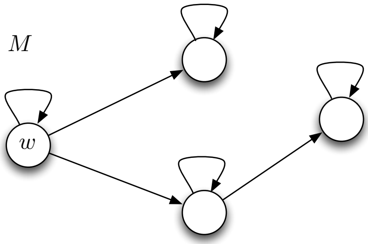
We say first First Order Property Φ corresponds to φ

This often gives rise to completeness: (let $C^{\Phi} = \{M \mid M \models \Phi\}$)

$$C^{\Phi} \models \psi \text{ iff } Ax + \varphi \vdash \psi$$

Example: Reflexivity

$$\forall x\ Rxx$$



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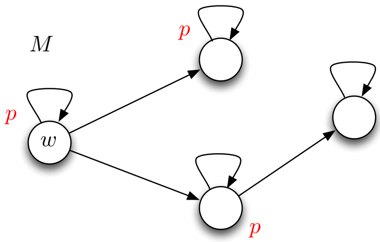

$$M, w \models \Box p$$

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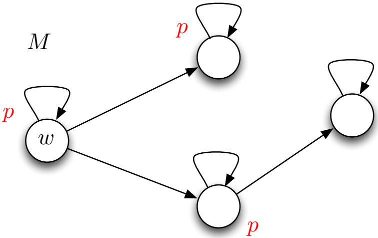

$$M, w \models \Box p \rightarrow p$$

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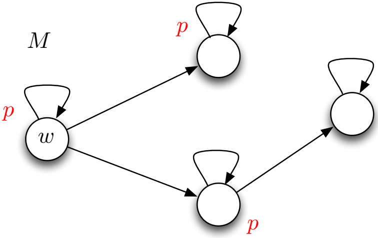
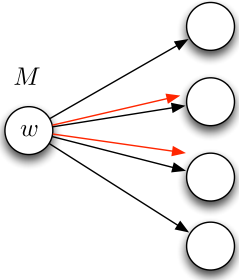


$$M, w \models \Box p \to p$$

$$M, w \models \Box \varphi \to \varphi$$

Example: Inclusion

$$\forall y\,(R_{a}xy \Rightarrow R_{b}xy)$$



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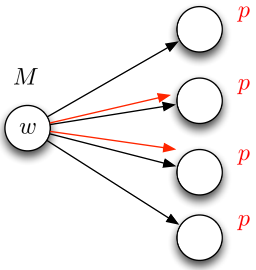

$$M, w \models [b]p$$

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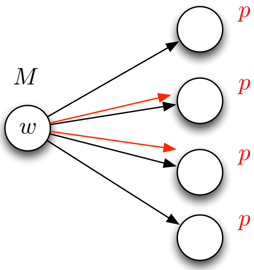

$$M, w \models [b]p \rightarrow [a]p$$

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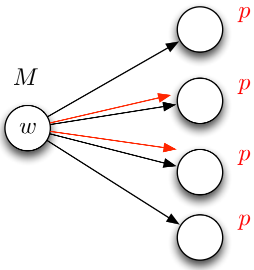

$$M, w \models [b]p \rightarrow [a]p$$

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- She knows at least as much as he does: $\varphi = K_h\chi \rightarrow K_s\chi$
- adding φ to Ax implies that $K_h(K_h\chi \rightarrow K_s\chi)$, that $\Box(K_h\chi \rightarrow K_s\chi)$ and $C(K_h\chi \rightarrow K_s\chi)$

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- but we cannot say: $M, w \models \forall\varphi(K_a\varphi \rightarrow K_b\varphi)$.
- we cannot say $M, w \models \forall\varphi(B_a\varphi \rightarrow \varphi) \wedge \neg B_b\forall\varphi(B_a\varphi \rightarrow \varphi)$.

How about local properties?

Wouldn't it be nice to be able to say:

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- John knows that she knows everything he knows, but Sarah does not know that
- If John's beliefs are correct, then so must Mary's be

Our models

$M = \langle W, R, I, V \rangle$. Then we define, for $\varphi \in \mathcal{L}(A, \pi, \rho)$:

$$
\begin{array}{lll}
M, w \models p & \text{iff} & w \in V(p) \\
M, w \models \neg\phi & \text{iff} & M, w \not\models \phi \\
M, w \models \phi \wedge \psi & \text{iff} & M, w \models \phi \text{ and } M, w \models \psi \\
M, w \models [a]\phi & \text{iff} & \text{for all } v \text{ if } R_a wv, \text{ then } M, v \models \phi
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$\boxdot(\vec{a}) = a \geq b$, and $I(\boxdot(\vec{a}))(w) = \forall v(R_b wv \Rightarrow R_a wv)$

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$\boxdot(\vec{a}) = Refl(a)$, and $I(\boxdot(\vec{a}))(w) = R_a ww$

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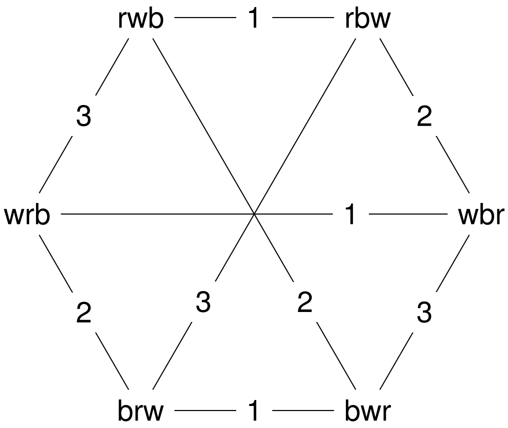
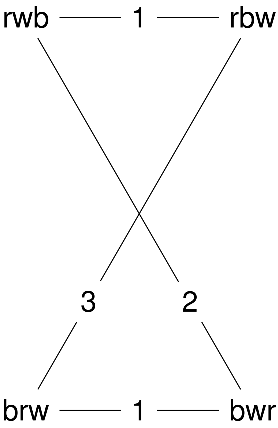
Talk: focus on $\boxdot(\vec{a}) = a \geq b$, property $K_a\chi \rightarrow K_b\chi$ and $R_a \supseteq R_b$ but our results are more general

Example: cards



Player 1 does not have white



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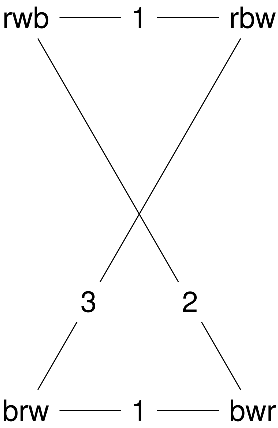

$M, rwb \models (1 \geq 3)$ and hence $M, rwb \models K_1\psi \rightarrow K_3\psi \ \ \forall\psi$

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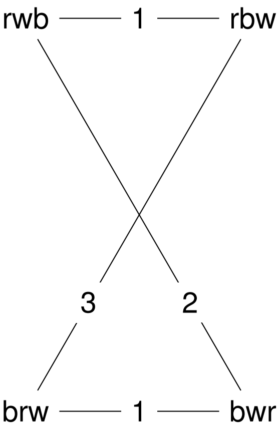
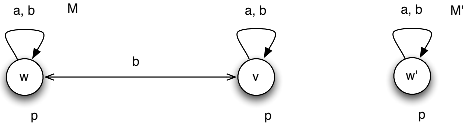


$$M, rwb \models (1 \geq 3) \wedge (2 \geq 3) \wedge \neg K_1(1 \geq 3)$$

Bisimulation



M, w and M', w' bisimulate, yet

$$M, w \models \neg(a \geq b) \text{ but } M', w' \models a \geq b$$

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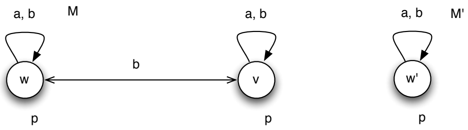

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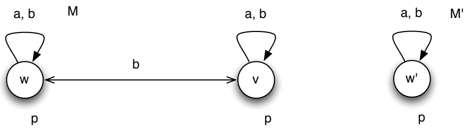

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Likewise, unraveling does not preserve $a \geq b$:
On every unravelled model, $\neg(a \geq b) \wedge \neg(b \geq a)$ holds.

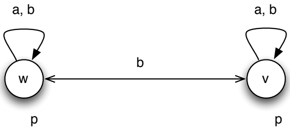
Axiom

- $(a \geq b) \rightarrow (K_a\varphi \rightarrow K_b\varphi)$

Axiom

- $(a \succeq b) \rightarrow (K_a\varphi \rightarrow K_b\varphi)$

The other direction is not valid:



$$M, w \models K_a\varphi \rightarrow K_b\varphi \wedge \neg(a \succeq b)$$

The Idea

Suppose $M, w \models \neg(a \geq b)$

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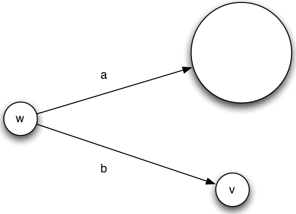

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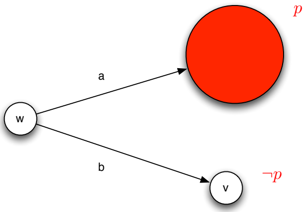
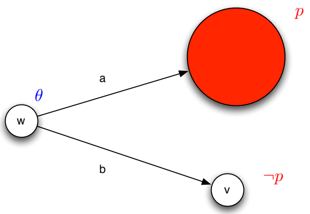


If $\neg(a \geq b)$ is satisfiable, then so is $K_a p \wedge \neg K_b p$

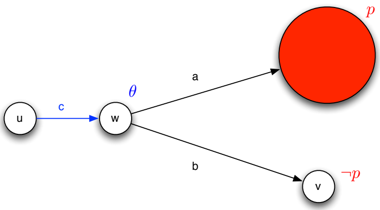
The Idea



If $\theta \wedge \neg(a \geq b)$ is satisfiable, then so is $\theta \wedge K_a p \wedge \neg K_b p \quad (p \notin \theta)$

The Idea

Suppose $M, u \models M_c(\theta \wedge \neg(a \geq b))$



If $M_c(\theta \wedge \neg(a \geq b))$ is satisfiable, then so is $M_c(\theta \wedge K_a p \wedge \neg K_b p)$ $(p \notin \theta)$

Rule

- The idea was:
  If $\langle s \rangle \neg(a \succeq b)$ is satisfiable, then so is $\langle s \rangle (K_a p \wedge \neg K_b p)$

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  If $\langle s \rangle \neg (a \succeq b)$ is satisfiable, then so is $\langle s \rangle (K_a p \wedge \neg K_b p)$
- If $\nvdash \neg \langle s \rangle \neg (a \succeq b)$, then $\nvdash \neg \langle s \rangle (K_a p \wedge \neg K_b p)$

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- If $\vdash \neg \langle s \rangle (K_a p \wedge \neg K_b p)$ then $\vdash \neg \langle s \rangle \neg (a \succeq b)$

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- If $\vdash \neg\langle s\rangle(K_a p \wedge \neg K_b p)$ then $\vdash \neg\langle s\rangle\neg(a \succeq b)$
- If $\vdash [s](K_a p \rightarrow K_b p)$, then $\vdash [s](a \succeq b)$.

Axiomatisation (of *Sup*)

Prop	All instances of propositional tautologies
K	$[a](\phi \rightarrow \psi) \rightarrow ([a]\phi \rightarrow [a]\psi)$
Ax_{Sup}	$(a \geq b) \rightarrow ([a]\varphi \rightarrow [b]\varphi)$
MP	From $\phi \rightarrow \psi$ and ϕ, infer ψ
Nec	From ϕ, infer $[a]\phi$
US	From φ infer $\varphi[\psi/p]$.
R_{Sup}	From $[s]([a]p \rightarrow [b]p)$, infer $[s](a \geq b)$, where $\vec{p}$ does not occur in s.

Properties of *Sup*

- $a \geq a$
- $(a \geq b) \wedge (b \geq c) \rightarrow (a \geq c)$
- $((a \geq b) \wedge \neg(b \geq a)) \rightarrow \langle a \rangle \top$

Axiomatisation (of *Sup* in *S5*)

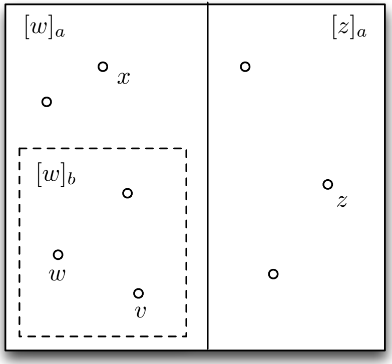
Prop	All instances of propositional tautologies
K	$K_a(\phi \rightarrow \psi) \rightarrow (K_a\phi \rightarrow K_a\psi)$
T	$K_a\varphi \rightarrow \varphi$
5	$\neg K_a\varphi \rightarrow K_a\neg K_a\varphi$
Ax$_{Sup}$	$(a \geq b) \rightarrow (K_a\varphi \rightarrow K_b\varphi)$
MP	From $\phi \rightarrow \psi$ and ϕ, infer ψ
Nec	From ϕ, infer $K_a\phi$
US	From φ infer $\varphi[\psi/p]$.
R$_{Sup}$	From $[s](K_ap \rightarrow K_bp)$, infer $[s](a \geq b)$, where p does not occur in s.

Consequences in $S5_m$

1. $a \geq b \leftrightarrow K_b(a \geq b)$
2. $a \succ b \leftrightarrow K_b(a \succ b)$
3. $\neg(a \geq b) \leftrightarrow K_b \neg(a \geq b)$
4. $K_b(a \geq b) \vee K_b \neg(a \geq b)$

Consequences in $S5_m$

$$a \succeq b \leftrightarrow K_b(a \succeq b)$$



Our Result

- add a symbol $\boxdot(\vec{a})$
- first order property $I(\boxdot(\vec{a}))(w)$
- corresponding formula $\theta_{\boxdot}(\vec{a}, \vec{p})$ for $I(\boxdot(\vec{a}))$

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Our Result

- add a symbol $\boxdot(\vec{a})$ $\quad (a \geq b)$
- first order property $I(\boxdot(\vec{a}))(w)$ $\quad R_a wv \Rightarrow R_b wv$
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Our Result

- add a symbol $\boxdot(\vec{a})$ $(a \geq b)$
- first order property $I(\boxdot(\vec{a}))(w)$ $R_a wv \Rightarrow R_b wv$
- corresponding formula $\theta_{\boxdot}(\vec{a}, \vec{p})$ for $I(\boxdot(\vec{a}))$ $K_b \chi \to K_a \chi$

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We get a sound and complete axiomatisation:

- add axiom $\boxdot(\vec{a}) \to \theta_{\boxdot}(\vec{a},\vec{p})$
- add inference rule: If $\vdash [s](\theta_{\boxdot}(\vec{a},\vec{p}))$, then $\vdash [s](\boxdot(\vec{a}))$
- $$\frac{\text{global modal logic : Sahlqvist formulas}}{\text{local modal logic : r-persistent formulas}}$$

Comparing Logics

Let $L_1 = \langle \Phi_1, \models_1, \mathbb{M} \rangle$ and $L_2 = \langle \Phi_2, \models_2, \mathbb{M} \rangle$ be two logics.

G. Gogic, C. Papadimitriou, B. Selman and H. Kautz, The comparative linguistics of knowledge representation

It is often the case that two logics are equally expressive, and either have similar computational complexity properties, or their respective complexities are so high that the differences almost meaningless in practical situations.

Comparing Logics: Succinctness

They propose to look at representational succinctness

Intuitively, if we are interested in some particular semantic property Q that is expressible with formulae φ_1 and φ_2 from two formalisms L_1 and L_2 respectively, we can ask if there is a significant difference in the lengths of φ_1 and φ_2, i.e. whether one of them is more succinct than the other.

Working definition of succinctness

Let $L_1 = \langle \Phi_1, \models_1, \mathbb{M} \rangle$ and $L_2 = \langle \Phi_2, \models_2, \mathbb{M} \rangle$ be two logics and let f be a strictly increasing polynomial.

Definition

L_1 is exponentially more succinct than L_2 on $\mathbb{M}$, written $L_1 \leq^{EXP}_{\mathbb{M}} L_2$, if for every $n \in \mathbb{N}$, there are two formulae $\alpha_n \in \Phi_1$ and $\beta_n \in \Phi_2$ satisfying the properties:

1. $|\alpha_n| \leq f(n)$ while $|\beta_n| \geq 2^{f(n)}$;
2. β_n is the shortest formula in Φ_2 with $\alpha_n \equiv_{\mathbb{M}} \beta_n$.

Succinctness: some Observations

1. $L_1 \preceq^{EXP}_{\mathbb{M}} L_2$ and $L_2 \preceq^{EXP}_{\mathbb{M}} L_1$ can be true simultaneously
   (We will see an example)

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This is because in $\mathbb{N}$, there may be more candidates that are equivalent to β_n.
3. It may even be that L_2 is more expressive than L_1, while still $L_1 \leq^{EXP}_{\mathbb{M}} L_2$!
(We will see an example)

Models

Given $\Sigma = \langle A, I \rangle$, we define $M = \langle W, R, V \rangle$ and

Definition

$(M, w) \models [i]\psi$	iff	for all v, $wR_i v$ implies $(M, v) \models \psi$;
$(M, w) \models [\exists_\Gamma]\psi$	iff	there is an $i \in \Gamma$ s.t. $(M, w) \models [i]\psi$;
$(M, w) \models [\forall_\Gamma]\psi$	iff	for all $i \in \Gamma$, $(M, w) \models [i]\psi$;
$(M, s) \models [\cap_\Gamma]\psi$	iff	for all t with $s \cap_{\gamma \in \Gamma} R_\gamma t, (M, t) \models \psi$;
$(M, w) \models [\psi_1]\psi_2$	iff	If $(M, w) \models \psi_1$, then $(M|_{\psi_1}, w) \models \psi_2$.

Logics

- The set Φ_{ML} of formulae of Multimodal Logic ML consists of

$$\varphi := p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid [i]\varphi$$

- Similarly, we talk about
  $[\forall_\Gamma]$ML and $[\exists_\Gamma]$ML and $[\cap_\Gamma]$ML and $[\varphi]$ML
- They are all equally expressive, except for $[\cap_\Gamma]$ML

Logics

$[\forall_\Gamma]\varphi$ with $\Gamma \subseteq I$ would mean:

1. (epistemically) 'everybody in Γ knows φ'
2. (dynamically) 'after every execution of any program form Γ, φ' (demonic non-deterministic choice)
3. (in description logic) it adds role disjunctions, like in $\forall(brother \sqcup sister).happy$

Our Concern

$$L \preceq_{\mathbb{M}}^{EXP} ML$$

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How do we know β_n are shortest ML formulae equivalent to α_n?

Formula Size Games (Adler & Immerman)

A one-person formula size game (*FSG*) on two sets of pointed models $\mathbb{A}$ and $\mathbb{B}$ is played as follows. During the course of the game, a game tree is constructed in such a way that each node is labeled with a pair $\langle \mathbb{C}, \mathbb{D} \rangle$ of sets of pointed models and one symbol from the set $\Sigma = \{p, \neg, \vee, [i]\}$. A node labeled with the pair $\langle \mathbb{C}, \mathbb{D} \rangle$ is denoted $\langle \mathbb{C} \circ \mathbb{D} \rangle$.
A node in the tree can be declared either open or closed.
Moves can be played only at open nodes. The game begins with the root of the game tree $\langle \mathbb{A} \circ \mathbb{B} \rangle$ which is declared "open".

Formula Size Games: Main Property

Theorem

Spoiler can win the FSG starting at $\langle \mathbb{A} \circ \mathbb{B} \rangle$ in n moves if and only if there is a formula $\phi \in \Phi_{ML}$ such that $\mathbb{A} \models \phi$, $\mathbb{B} \models \neg\phi$, and $|\phi| = n$.

Moves in FSGs

Spoiler can play one of the following at an open node $\langle \mathbb{C} \circ \mathbb{D} \rangle$

atomic-move: Spoiler chooses a propositional symbol p such that $\mathbb{C} \models p$ and $\mathbb{D} \models \neg p$. The node is declared closed and labeled with the symbol p.

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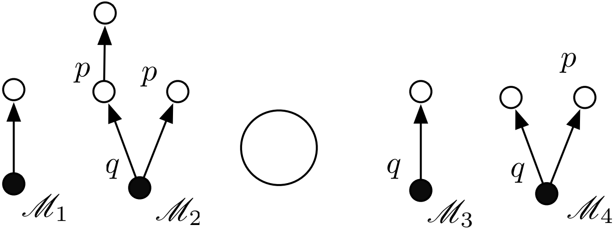
or-move: Spoiler labels the node with the symbol $\vee$ and chooses two subsets $\mathbb{C}_1 \subseteq \mathbb{C}$ and $\mathbb{C}_2 \subseteq \mathbb{C}$ such that $\mathbb{C} = \mathbb{C}_1 \cup \mathbb{C}_2$. Two new open nodes are added to the tree as successors to the node $\langle \mathbb{C} \circ \mathbb{D} \rangle$, namely $\langle \mathbb{C}_1 \circ \mathbb{D} \rangle$ and $\langle \mathbb{C}_2 \circ \mathbb{D} \rangle$.

Moves in FSGs

Spoiler can play one of the following at an open node $\langle \mathbb{C} \circ \mathbb{D} \rangle$

[i]-move: Spoiler labels the node with the symbol [i] and, for each pointed model $(D, v) \in \mathbb{D}$, he chooses a pointed model (D, v') such that vR_iv' (if for some $(D, v) \in \mathbb{D}$ this is not possible, Spoiler cannot play this move). All these new pointed models are collected in a set $\mathbb{D}_1$. A set of models $\mathbb{C}_1$ is then constructed as follows. For each pointed model $(C, w) \in \mathbb{C}$, all the possible pointed models (C, w') such that wR_iw' are added to $\mathbb{C}_1$. If for some (C, w), the point w does not have an R_i-successor, nothing is added to $\mathbb{C}_1$ for the pointed model (C, w). A new open node $\langle \mathbb{C}_1 \circ \mathbb{D}_1 \rangle$ is added as a successor to the node $\langle \mathbb{C} \circ \mathbb{D} \rangle$. In this case, we also say that Spoiler has played an index-move.

FSGs: Example $\neg q \vee [i]p$



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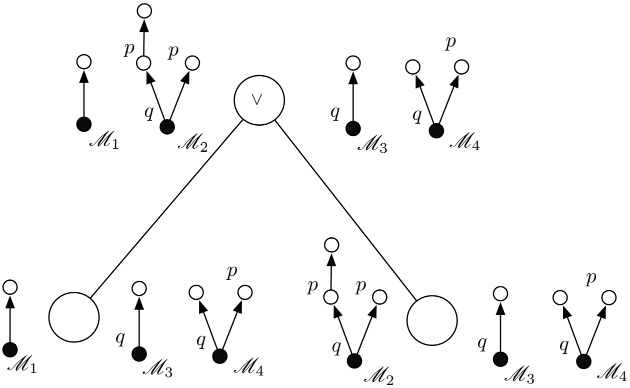

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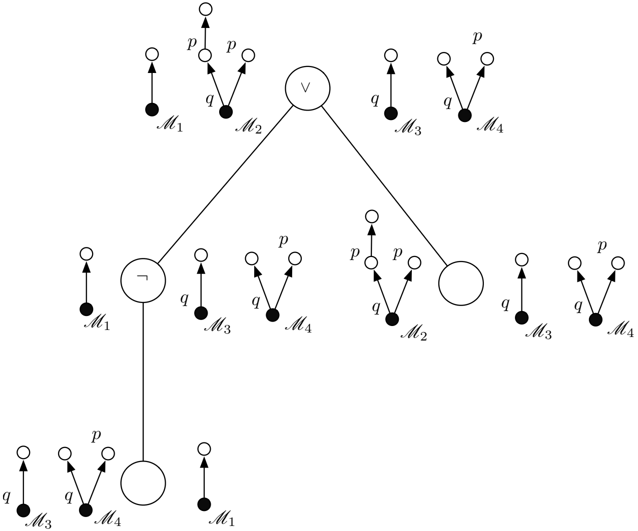

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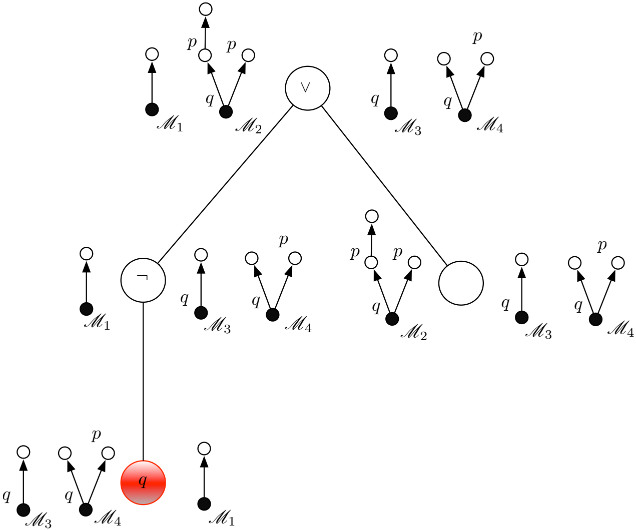

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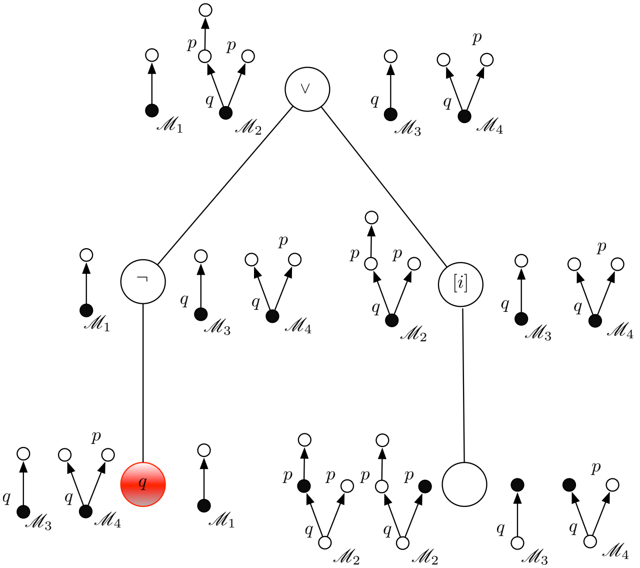

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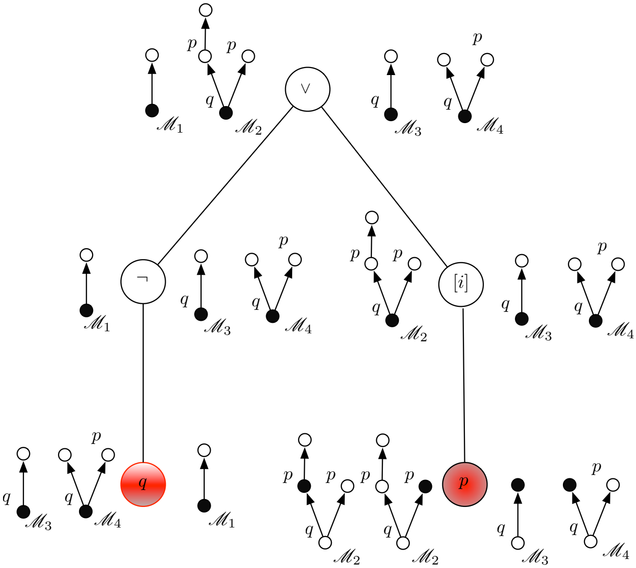

General Models

Let $I \supseteq \Gamma = \{a, b\}$ and $A \supseteq \{p\}$.

Theorem

$[\forall_\Gamma]ML \preceq_{\mathbf{K}}^{EXP} ML$.

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1. $\alpha_n = \neg[\forall_{\{a,b\}}]^n \neg p$;
2. $\beta_1 = \langle a \rangle p \vee \langle b \rangle p$ and $\beta_{n+1} = \langle a \rangle \beta_n \vee \langle b \rangle \beta_n$.

FSGs

We are looking for sets of models $\mathbb{A}^n$ and $\mathbb{B}^n$ such that

1. $\mathbb{A}^n \models \beta_n$ and $\mathbb{B}^n \models \neg\beta_n$.
2. the shortest way to win $\langle \mathbb{A}^n \circ \mathbb{B}^n \rangle$ takes 2^n moves.

$n = 3$

$$\alpha_3 = \neg[\forall_{\{a,b\}}]^3 \neg p$$

$$\beta_1 = \langle a \rangle p \vee \langle b \rangle p$$
$$\beta_2 = \langle a \rangle(\langle a \rangle p \vee \langle b \rangle p) \vee \langle b \rangle(\langle a \rangle p \vee \langle b \rangle p)$$
$$\beta_3 = \langle a \rangle(\langle a \rangle(\langle a \rangle p \vee \langle b \rangle p) \vee \langle b \rangle(\langle a \rangle p \vee \langle b \rangle p)) \vee \langle b \rangle(\langle a \rangle(\langle a \rangle p \vee \langle b \rangle p) \vee \langle b \rangle(\langle a \rangle p \vee \langle b \rangle p))$$

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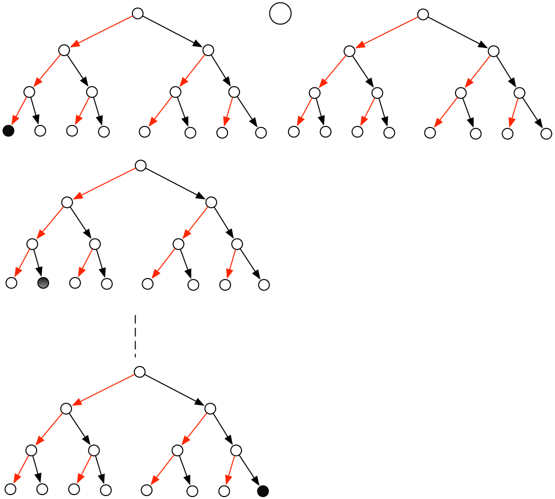
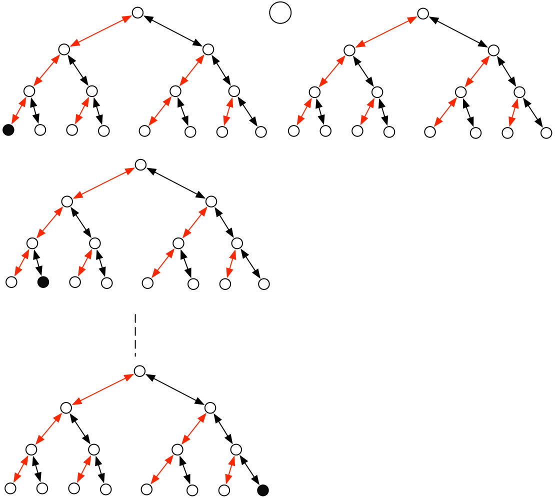


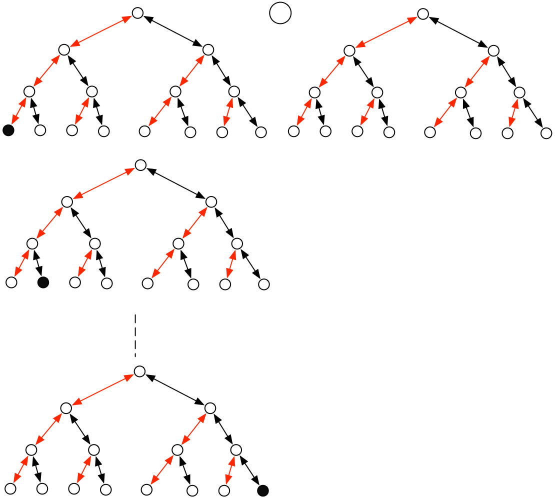
Does this work in **S5**?

$n = 3$. $\alpha_3 = \neg[\forall_{\{a,b\}}]^3 \neg p$



Does this work in **S5**?

$n = 3$. $\alpha_3 = \neg[\forall_{\{a,b\}}]^3 \neg p$ Now: $\beta'_3 = \langle a \rangle \langle b \rangle \langle a \rangle \langle b \rangle \langle a \rangle \langle b \rangle p$



Results

Theorem

1. *Suppose that* $|P| \geq 1$ *and* $|I| \geq 2$. *Then*

$$[\exists]ML \leq^{EXP}_{\mathbb{K}} [\forall]ML \;\&\; [\forall]ML \leq^{EXP}_{\mathbb{K}} [\exists]ML.$$

2. *If* $|P| \geq 3$ *and* $|I| \geq 4$, *then*

$$[\exists]ML \leq^{EXP}_{\mathbb{S}5} ML; \; [\forall]ML \leq^{EXP}_{\mathbb{S}5} ML; \; [\varphi]ML \leq^{EXP}_{\mathbb{S}5} ML.$$

More results

Theorem

1. $[\varphi]ML \preceq_{\mathbb{K}}^{EXP} [\cap]ML$;
2. $[\varphi]ML \preceq_{\mathbb{K}}^{EXP} [\forall]ML$;
3. $[\varphi]ML \preceq_{\mathbb{K}}^{EXP} [\exists]ML$;
4. $[\cap]ML \preceq_{\mathbb{K}}^{EXP} [\forall]ML$;
5. $[\forall]ML \preceq_{\mathbb{K}}^{EXP} [\cap]ML$;
6. $[\varphi]ML \preceq_{\mathbb{K}}^{EXP} [\cap, \exists]ML$;
7. $[\forall]ML \preceq_{\mathbb{K}}^{EXP} [\cap, \exists]ML$

More results

Theorem

1. $[\varphi]ML \preceq_{\mathbb{K}}^{EXP} [\cap]ML$;
2. $[\varphi]ML \preceq_{\mathbb{K}}^{EXP} [\forall]ML$;
3. $[\varphi]ML \preceq_{\mathbb{K}}^{EXP} [\exists]ML$;
4. $[\cap]ML \preceq_{\mathbb{K}}^{EXP} [\forall]ML$;
5. $[\forall]ML \preceq_{\mathbb{K}}^{EXP} [\cap]ML$;
6. $[\varphi]ML \preceq_{\mathbb{K}}^{EXP} [\cap, \exists]ML$;
7. $[\forall]ML \preceq_{\mathbb{K}}^{EXP} [\cap, \exists]ML$

Conclusion

- Complete logic for "knowing locally at least as much as".
- This notion was generalised to other modal properties.
- Succinctness: tight results? (# atoms and # indices)
- $[\varphi]ML \leq^{EXP}_{S5} ML$ solves a problem of Lutz 2006
- Is $[\exists_\Gamma]ML$ or $[\forall_\Gamma]ML$ more succinct than $[\varphi]ML$?
- How widely applicable are the Adler-Immerman games?